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Note on the Carleman's inequality and Hardy's inequality[☆]

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ABSTRACT

In this article, using the properties of power mean and induction, new strengthened Carleman's inequality and Hardy's inequality are obtained. We also give an answer to the conjectures proposed by X. Yang in the literature Yang (2001) [5].

1. Introduction

The following Carleman's inequality is well known (see [1])

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n}<e\sum_{n=1}^{\infty}a_n \tag{1}$$

where $a_n\geq 0$, $n=1,2,\ldots$, and $0<\sum_{n=1}^{\infty}a_n<\infty$.

By Hardy (see, [1, Theorem 349]), the Carleman's inequality was generalized as follows:

$$\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n}<e\sum_{n=1}^{\infty}\lambda_n a_n \tag{2}$$

where $a_n\geq 0$, $\lambda_n>0$, $\Lambda_n=\sum_{m=1}^{n}\lambda_m$, $n=1,2,\ldots$, and $0<\sum_{n=1}^{\infty}\lambda_n a_n<\infty$.

In recent years, Yang [2], Yan and Sun [3], Xie and Zhong [4], Yang [5], and Dragomir and Ho Kim [6] gave some distinct improvements of (1) or (2), respectively.

Yang [5] proposed the following conjectures:

Conjecture 1. *Let $x\geq 0$. Then the following equality holds*

$$\left(1+\frac{1}{x}\right)^x=e\left[1-\sum_{k=1}^{\infty}\frac{b_k}{(1+x)^k}\right],$$

where $b_1=\frac{1}{2}$, $b_2=\frac{1}{24}$, $b_3=\frac{1}{48}$, $b_4=\frac{73}{5760}$, $b_5=\frac{11}{1280}$ and $b_6=\frac{1945}{580608}$.

Conjecture 2. *Is $b_k>0$, for $k\geq 7$?*

In this note, we give an answer on these conjectures. New, strengthened Carleman's and Hardy's inequalities are also obtained.

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2. Some lemmas

We will use the following lemmas.

Lemma 2.1 ([6]). *Let $x > 0$, then*

$$\left(1+\frac{1}{x}\right)^x = e\left[1-\sum_{k=1}^{\infty}\frac{b_k}{(1+x)^k}\right], \tag{3}$$

where $b_1 = \frac{1}{2}$, $b_{n+1} = \frac{1}{n+1}\left(\frac{1}{n+2} - \sum_{k=1}^{n}\frac{b_k}{n+2-k}\right)$, $n = 1, 2, \ldots$.

Lemma 2.2. *Let $x > 0$, $m \in \mathbb{N}$, then*

$$\left(1+\frac{1}{x}\right)^x < e\left[1-\sum_{k=1}^{m}\frac{b_k}{(1+x)^k}\right], \tag{4}$$

where $b_1 = \frac{1}{2}$, $b_{n+1} = \frac{1}{n+1}\left(\frac{1}{n+2} - \sum_{k=1}^{n}\frac{b_k}{n+2-k}\right)$, $n = 1, 2, m-1$.

Proof. By Lemma 2.1, we only prove that $b_n > 0$, $n \in \mathbb{N}$. By induction, because $b_1 = \frac{1}{2} > 0$, we will prove that: if $b_t > 0$, as $n = t-1$, $t \in \mathbb{N}$ and $t \geq 2$, then $b_{t+1} > 0$, as $n = t$.

In fact, when $n = t$ ($t \in \mathbb{N}$ and $t \geq 2$),

$$\begin{aligned}
b_{t+i} &= \frac{1}{t+1}\left(\frac{1}{t+2} - \sum_{k=1}^{t}\frac{b_k}{t+2-k}\right)\\
&= \frac{1}{t+1}\left(\frac{1}{t+2} - \sum_{k=1}^{t-1}\frac{b_k}{t+2-k} - \frac{b_t}{2}\right)\\
&= \frac{1}{t+1}\left[\left(\frac{1}{t+2} - \frac{1}{2t(t+1)}\right) - \sum_{k=1}^{t-1}\left(\frac{1}{t+2-k} - \frac{1}{2t(t+1-k)}\right)b_k\right]\\
&= \frac{1}{t+1}\left[\left(\frac{t+1}{t+2} - \frac{1}{2t}\right)\frac{1}{t+1} - \sum_{k=1}^{t-1}\left(\frac{t+1-k}{t+2-k} - \frac{1}{2t}\right)\frac{b_k}{t+1-k}\right]\\
&> \frac{1}{t+1}\left[\left(\frac{t+1}{t+2} - \frac{1}{2t}\right)\frac{1}{t+1} - \left(\frac{t+1}{t+2} - \frac{1}{2t}\right)\sum_{k=1}^{t-1}\frac{b_k}{t+1-k}\right]\\
&= \frac{t}{t+1}\left(\frac{t+1}{t+2} - \frac{1}{2t}\right)\left[\frac{1}{t}\left(\frac{1}{t+1} - \sum_{k=1}^{t-1}\frac{b_k}{t+1-k}\right)\right]\\
&= \frac{t}{t+1}\left(\frac{t+1}{t+2} - \frac{1}{2t}\right)b_t.
\end{aligned}$$

By induction, we have $b_n > 0$, $n \in \mathbb{N}$. The lemma is proved. □

Remark 2.3. Lemmas 1.1 and 2.2 have given an answer of Conjectures 1 and 2.

By Lemma 2.2, we have the following corollary.

Corollary 2.4. *Let $a_n \geq 0$, $m \in \mathbb{N}$, $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n}\lambda_n$ satisfying $\Lambda_n \geq 1$, then*

$$\left(1+\frac{\lambda_n}{\Lambda_n}\right)^{\Lambda/\lambda} < e\left[1-\sum_{k=1}^{m}\frac{\lambda_k b_k}{(\lambda_n+\Lambda_n)^k}\right], \tag{5}$$

where $b_1 = \frac{1}{2}$, $b_{n+1} = \frac{1}{n+1}\left(\frac{1}{n+2} - \sum_{k=1}^{n}\frac{b_k}{n+2-k}\right)$, $n = 1, 2, m-1$.

3. Main results

Now we introduce the main results.

Theorem 3.1. *Let* $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_n$ *satisfying* $\Lambda_n \ge 1$, $a_n \ge 0 (n \in \mathbb{N})$, $0 < p \le 1$, $m \in \mathbb{N}$ *and* $0 < \sum_{n=1}^{m} \lambda_n a_n < \infty$. *Then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < \frac{e^p}{p} \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{m} \frac{\lambda_n^k b_k}{(\lambda_n + \Lambda_n)^k}\right)^p \lambda_n a_n^p \Lambda_n^{p-1} \left(\sum_{s=1}^{n} \lambda_s (c_s a_s)^p\right)^{(1-p)/p} \tag{6}$$

where $c_s^{\lambda_n} = \Lambda_{n+1}^{\Lambda_n} / \Lambda_n^{\Lambda_{n-1}}$, $b_1 = \frac{1}{2}$, $b_{n+1} = \frac{1}{n+1}\left(\frac{1}{n+2} - \sum_{k=1}^{n} \frac{b_k}{n+2-k}\right)$, $n = 1, 2, m - 1$.

Proof. By the power mean inequality, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} &= \sum_{n=1}^{\infty} \lambda_{n+1} \frac{(c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \cdots ((c_n a_n)^{\lambda_n/\Lambda_n})^{1/\Lambda_n}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \\ &\le \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}} \left(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda (c_m a_m)^p\right)^{1/p} \end{aligned} \tag{7}$$

For $c_m > 0$, $m = 1, 2, \ldots, n$.

By using the Hardy–Littlewood inequality (see [7,8]), we have

$$\begin{aligned}\left(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p\right)^{1/p} &\le \frac{1}{\Lambda_n} \left(\sum_{m=1}^{n} \lambda_m (c_m a_m)^p\right)^{1/p} \\ &\le \frac{1}{p\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p \left(\sum_{s=1}^{m} \lambda_s (c_s a_s)^p\right)^{(1-p)/p}. \end{aligned} \tag{8}$$

For $\Lambda_n \ge 1$, $0 < p \le 1$.

Choosing $c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} = \Lambda_{n+1}^{\Lambda_n} (n \in \mathbb{N})$, and setting $\Lambda_0 = 0$, from $\lambda_{n+1} \le \lambda_n$, it follows that

$$c_n = \left[\frac{\Lambda_{n+1}^{\Lambda_n}}{\Lambda_n^{\Lambda_{n-1}}}\right]^{1/\lambda_n} = \left[1 + \frac{\lambda_{n+1}}{\Lambda_n}\right]^{\Lambda_n/\lambda_n} \cdot \Lambda_n \le \left[1 + \frac{\lambda_n}{\Lambda_n}\right]^{\Lambda_n/\lambda_n} \cdot \Lambda_n. \tag{9}$$

From (7)–(9), it follows that

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} \le \frac{1}{p} \sum_{m=1}^{\infty} \left[1 + \frac{\lambda_m}{\Lambda_m}\right]^{p\Lambda_m/\lambda_m} \cdot \lambda_m \Lambda_m^{p-1} a_m^p \left(\sum_{s=1}^{m} \lambda_s (c_s a_s)^p\right)^{(1-p)/p}.$$

Hence by the above inequality and Lemma 2.2, we have

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < \frac{e^p}{p} \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{m} \frac{\lambda_n^k b_k}{(\lambda_n + \Lambda_n)^k}\right)^p \lambda_n a_n^p \Lambda_n^{p-1} \left(\sum_{s=1}^{n} \lambda_s (c_s a_s)^p\right)^{(1-p)/p}.$$

Thus Theorem 3.1 is proved. □

Setting $p = 1$ in Theorem 3.1, then we get an extension of the strengthened Hardy's inequality as following.

Corollary 3.2. *Let* $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_n$ *satisfying* $\Lambda_n \ge 1$, $a_n \ge 0 (n \in \mathbb{N})$, $m \in \mathbb{N}$ *and* $0 < \sum_{n=1}^{m} \lambda_n a_n < \infty$. *Then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{m} \frac{\lambda_n^k b_k}{(\lambda_n + \Lambda_n)^k}\right) \lambda_n a_n \tag{10}$$

where $c_s^{\lambda_n} = \Lambda_{n+1}^{\Lambda_n} / \Lambda_n^{\Lambda_{n-1}}$, $b_1 = \frac{1}{2}$, $b_{n+1} = \frac{1}{n+1}\left(\frac{1}{n+2} - \sum_{k=1}^{n} \frac{b_k}{n+2-k}\right)$, $n = 1, 2, m - 1$.

Setting $\lambda_n = 1$ *in* Theorem 3.1, *then we also get an extension of the strengthened Carleman's inequality as the following.*

Corollary 3.3. *Let* $a_n \ge 0 (n \in \mathbb{N})$, $0 < p \le 1$, $m \in \mathbb{N}$ *and* $0 < \sum_{n=1}^{m} \lambda_n a_n < \infty$. *Then*

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < \frac{e^p}{p} \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{m} \frac{\lambda_n^k b_k}{(1 + n)^k}\right)^p a_n^p n^{p-1} \left(\sum_{s=1}^{n} (c_s a_s)^p\right)^{(1-p)/p} \tag{11}$$

where $c_s = (1 + 1/s)^s \cdot s$, $b_1 = \frac{1}{2}$, $b_{n+1} = \frac{1}{n+1}\left(\frac{1}{n+2} - \sum_{k=1}^{n} \frac{b_k}{n+2-k}\right)$, $n = 1, 2, m - 1$.

Similarly to the proof of Theorem 3.1, we can prove the generalized version of Hardy's inequality as the following theorem:

Theorem 3.4. *Let* $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_n$ *satisfying* $\Lambda_n \geq 1$, $a_n \geq 0 (n \in \mathbb{N})$, $m \in \mathbb{N}$ *and* $0 < \sum_{n=1}^{m} \lambda_n a_n^t < \infty$ *for* $0 < p \leq t < \infty$, *Then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} < \frac{t e^{p/t}}{p} \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{m} \frac{\lambda_n^k b_k}{(\lambda_n + \Lambda_n)^k}\right)^{p/t} \lambda_n a_n^p \Lambda_n^{(p-t)/t} \left(\sum_{s=1}^{n} \lambda_s (c_s a_s)^p\right)^{(t-p)/p} \quad (12)$$

where $c_s^{t\lambda_n} = \Lambda_{n+1}^{\Lambda_n} / \Lambda_n^{\Lambda_{n-1}}$, $b_1 = \frac{1}{2}$, $b_{n+1} = \frac{1}{n+1}(\frac{1}{n+2} - \sum_{k=1}^{n} \frac{b_k}{n+2-k})$, $n = 1, 2, m - 1$.

The proof is almost the same as in proving Theorem 3.1. We here only need to note that

$$(\alpha_1^{q_1} \alpha_2^{q_2} \cdots \alpha_n^{q_n})^t \leq \left(\sum_{m=1}^{n} q_m a_m^p\right)^{t/p}, \quad p, t \geq 0$$

where $\alpha_m \geq 0$, $0 < p \leq t < \infty$, $q_{m>0}$, and $\sum_{m=1}^{n} q_m = 1 (m \in N)$.

Setting $p = 1$ in Theorem 3.4, then we get an extension of the strengthened Hardy's inequality as the following.

Corollary 3.5. *Let* $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_n$ *satisfying* $\Lambda_n \geq 1$, $a_n \geq 0 (n \in \mathbb{N})$, $m \in \mathbb{N}$ *and* $0 < \sum_{n=1}^{m} \lambda_n a_n^t < \infty$ *for* $1 \leq t < \infty$. *Then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{t/\Lambda_n} < t e^{1/t} \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{m} \frac{\lambda_n^k b_k}{(\lambda_n + \Lambda_n)^k}\right)^{1/t} \lambda_n a_n \Lambda_n^{(1-t)/t} \left(\sum_{s=1}^{n} \lambda_s (c_s a_s)\right)^{(t-1)}. \quad (13)$$

where $c_s^{t\lambda_n} = \Lambda_{n+1}^{\Lambda_n} / \Lambda_n^{\Lambda_{n-1}}$, $b_1 = \frac{1}{2}$, $b_{n+1} = \frac{1}{n+1}(\frac{1}{n+2} - \sum_{k=1}^{n} \frac{b_k}{n+2-k})$, $n = 1, 2, m - 1$.

Setting $\lambda_n = 1$ in Theorem 3.4, then we also get an extension of the strengthened Carleman's inequality as the following.

Corollary 3.6. *Let* $a_n \geq 0 (n \in \mathbb{N})$, $m \in \mathbb{N}$ *and* $0 < \sum_{n=1}^{m} a_n^t < \infty$ *for* $0 < p \leq t < \infty$, *Then*

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{t/n} < \frac{t e^{p/t}}{p} \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{m} \frac{b_k}{(1+n)^k}\right)^{p/t} a_n^p n^{(p-t)/t} \left(\sum_{s=1}^{n} (c_s a_s)^p\right)^{(t-p)/p} \quad (14)$$

where $c_s^t = (1 + 1/s)^s \cdot s$, $b_1 = \frac{1}{2}$, $b_{n+1} = \frac{1}{n+1}(\frac{1}{n+2} - \sum_{k=1}^{n} \frac{b_k}{n+2-k})$, $n = 1, 2, m - 1$..

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